Incorporating Quadratic Scale Curves in Inverse Demand Systems

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Introduction

- Inverse demand systems may be more appropriate than direct demand systems for many problems arising in agricultural economics where quantities are relatively fixed in the short run and prices adjust to clear markets

- Ryan and Wales (1999) develop demand systems augmented to allow for quadratic Engel curves. The additional non-linearities in income allow the models to provide a better fit to empirical data

- We may have a similar situation in the case of inverse demand systems: additional terms in scale may be necessary in order to better represent reality

Contributions

- Three new functional forms quadratic in scale are developed that maintain linearity as a special case
- These systems are estimated while maintaining concavity restrictions (although only NQID-QSS reported here)
- Rank reduction procedures of Diewert and Wales (1988) employed to obtain semiflexible versions of the models
- Estimates are obtained for the effects on welfare of placing restrictions on landings of South Atlantic finfish

Background

- Many specifications for inverse demand systems are derived from a representation of the consumer’s distance function. One parameterization (analogous to the Gorman polar form) is:

  (1) $D(u,\underline{q}) = a(\underline{q}) - ub(\underline{q})$

  where D is an an ordinal measure of ‘distance’,
  u is an arbitrary utility target,
  $\underline{q}$ is an n-vector of quantities,
  and $a(\underline{q})$ and $b(\underline{q})$ are linear homogeneous and concave functions

- A useful normalization is to assume that if $\underline{q}$ exactly generates u, then $D(u,\underline{q}) = 1$

- From (1) above, when $D(u,\underline{q}) = 1$, the distance function may be solved for u in terms of $\underline{q}$, yielding

(2) $$U(\underline{q}) = -\frac{(1-a(\underline{q}))}{b(\underline{q})}$$

- Applying the Shephard-Hannoch lemma to (1), and using (2) to eliminate u, the following price-dependent system is obtained:

(3) $$\pi_i = a_i(\underline{q}) + \frac{b_i(\underline{q})}{b(\underline{q})}[1-a(\underline{q})],\ i=1,\dots,n$$

where $\pi_i = p_i / y$ denotes the normalized price,
y denotes expenditure,
p_i denotes the nominal price,
and $a_i(\underline{q})$ and $b_i(\underline{q})$ are the ith first partial derivatives of $a(\underline{q})$ and $b(\underline{q})$

- Price equations of this sort are occasionally reported in the literature (Holt and Bishop), but (3) is a potentially restrictive specification because if $\underline{q}$ is scaled by an arbitrary constant, (1/τ), then π_i will be linear in τ - a linear scale curve.

- In direct demand systems, restricting demand functions to be linear in expenditure eliminates the possibility of luxury goods and inferior goods and this is overly restrictive for many goods

- Restricting inverse demand functions to be linear in scale implies the similarly implausible result that prices associated with consuming proportionately more of all goods in the bundle will change by the same amount for a given change in scale regardless of the size of the initial bundle

Quadratic Scale Systems

- Following Ryan and Wales, it may be shown that quadratic demand systems are generated from indirect utility functions of the form:

$$(4) \qquad \varphi(\underline{p}, y) = -\frac{g(\underline{p})}{y - f(\underline{p})} - h(\underline{p})$$

where $y = \Sigma p_k x_k$ is income or total expenditure, f and g are restricted to be homogeneous of degree one in prices, and h is homogeneous of degree zero in prices

- (4) may be easily rearranged to solve for the expenditure function, but we are interested in the distance function which corresponds to this expenditure function

- For a given level of utility u, a vector of consumption quantities $\underline{q} = (q_1,...,q_n)^{\mathrm{T}}$, and a representative consumer's indirect preference function $V(\underline{\pi})$, the distance function may be defined as

(6) $$D(u,\underline{q}) = \min_{\underline{\pi}}\{\underline{\pi}^T\underline{q} : V(\underline{\pi}) \geq u\}$$

- Distance functions are non-increasing in u and are increasing, homogeneous of degree one, and concave in $\underline{q}$.

- Intuitively, the distance function is the amount by which $\underline{q}$ must be divided to bring it on the indifference surface u

- Taking the first and second derivatives of the distance function yields compensated inverse demands and the Antonelli matrix, A, respectively. The Antonelli matrix is symmetric, negative semidefinite, and due to homogeneity, is at most of rank n-1

- A potentially useful specification for the distance function is simply:

(9) $$D(u,\underline{q}) = \frac{g(\underline{q})}{u + h(\underline{q})} + f(\underline{q}),$$

acquired through taking the distance function corresponding to the expenditure function derived from the indirect utility function in (4)

- As mentioned above, when quantities are such that utility level *u* is reached, the normalization $D(u,\underline{q})$ = 1is typically applied. The distance function may then be inverted to solve for the direct utility function

(10) $$U(\underline{q}) = \frac{g(\underline{q})}{1 - f(\underline{q})} - h(\underline{q})$$

- By applying the Shephard-Hannoch lemma to the direct utility function, compensated inverse demands of the form

$$\pi_i(\underline{q},u) = \frac{\partial d(\underline{q},u)}{\partial q_i} = \frac{(u+h(\underline{q}))g_i(\underline{q}) - g(\underline{q})h_i(\underline{q})}{(u+h(\underline{q}))^2} + f_i(\underline{q}) \tag{11}$$

are obtained, where f_i, g_i, and h_i are first partial derivatives of f, g, and h, respectively

- To acquire the uncompensated inverse demands, (10) is substituted into (11) to eliminate the unobservable u, giving us:

$$\pi_i(\underline{q}) = \frac{h_i(\underline{q})}{g(\underline{q})}[1-f(\underline{q})]^2 + \frac{g_i(\underline{q})}{g(\underline{q})}[1-f(\underline{q})] + f_i(\underline{q}). \tag{12}$$

- Notice that now if all quantities are scaled by the same factor $(1/\tau)$, then the resulting scale curves clearly include terms both linear and quadratic in τ

- Remaining theoretical issues involve choice of functional forms for f, g, and h.

The Normalized Quadratic Inverse Demand – Quadratic Scale System

- For the NQID-QSS, *f*, *g* and *h* are specified as follows:

$$(13) \qquad f(\underline{q}) = \sum_k q_k d_k,$$

$$(14) \qquad h(\underline{q}) = \sum_k a_k \ln q_k, \qquad \sum_{\mathrm{k}} a_k = 0,$$

$$(15) \qquad g(\underline{q}) = \sum_k q_k b_k + \frac{1}{2}\left(\frac{\sum_k \sum_j B_{kj} q_k q_j}{\sum_k \alpha_k q_k} \right).$$

- Note from (14) that if $a_k = 0 \; \forall \; k$, then $h_k(\underline{q}) = 0 \; \forall \; k$, giving us a direct way to test for linearity of scale curves

- Substituting (13) – (15) into (12), the normalized inverse demands may be written as

$$(16)\qquad \pi_i(q) = \frac{a_i}{q_i g}\Big[1-\sum_k q_k d_k\Big]^2 + \frac{b_i + \eta^{-1}\sum_k B_{ik} q_k - \frac{1}{2}\alpha_i \eta^{-2}\sum_k\sum_j B_{kj} q_k q_j}{g}\Big[1-\sum_k q_k d_k\Big] + d_i,$$

where $\eta = \sum_k \alpha_k q_k$; a_k , b_k , d_k , and B_{kj} are unknown parameters, and the $\alpha_k > 0$ are predetermined parameters, k, j = 1,...,n. We choose a reference vector of quantities $\underline{q}^* = (q_1^*,...,q_n^*)$. We then assume that the *nxn* matrix B satisfies the n restrictions

$$(17)\qquad B\underline{q}^* = 0, \qquad B = B^T.$$

- In addition, we assume that the following restrictions hold at the reference bundle $\underline{q}^*$:

$$\underline{\alpha}^T \underline{q}^* = 1, \qquad \underline{\alpha} \geq \underline{0}_n, \tag{18}$$

$$\underline{d}^T \underline{q}^* = 0, \tag{19}$$

$$\underline{b}^T \underline{q}^* = 1. \tag{20}$$

- There is no reason to believe that curvature requirements will be spontaneously satisfied by the estimated NQID-QSS, even at $\underline{q}^*$. Curvature may be imposed locally, however, through a Cholesky decomposition of the Antonelli matrix

- Using the Cholesky decomposition allows us to impose concavity on the distance function, but without affecting the flexibility properties of the inverse demand functions (Moschini; Ryan and Wales)

Empirical Application

- Inverse ex-vessel demands for finfish landed in the South Atlantic region of the U.S. are estimated as an illustration of the applicability of quadratic scale models

- Data compiled from NMFS data on monthly landings and total valuation of landings over the period from 1/80 – 12/96 (204 observations)

- Data for all reported species were aggregated into nine categories comprised of bluefish, dolphinfish, other finfish, flounder, groupers, scups, trout, snappers, and tilefish/triggerfish

- All quantities normalized to have unit mean. Therefore, without loss of generality, the reference point is chosen to be $\underline{q}^* = \underline{\iota}_n$ where $\underline{\iota}_n$ is the unit vector

- For estimation, we convert (16), our NQID-QSS equations, to share form by multiplying both sides by q_i and adding a stochastic disturbance term, giving us

$$(34)\qquad w_{it} = \frac{a_i}{g}\left[1-\sum_{k=1}^{n} q_{kt} d_k\right]^2 + \left(\frac{b_i q_{it} + \eta^{-1}\sum_{k=1}^{n} B_{ik} q_{it} q_{kt} - \frac{1}{2}\alpha_i q_{it}\eta^{-2}\underline{q}^T B \underline{q}^T}{g}\right)\left[1-\sum_{k=1}^{n} q_{kt} d_k\right] + d_i q_{it} + \nu_{it}$$

where $k = 1,\dots,9$, $n = 9$, and $t = 1,\dots,204$. The expression g refers to that given in (16). w_{it} denotes the expenditure share of the *i*th fish category in total fish sales and ν_{it} is an *iid* mean-zero error term.

- Because the contemporaneous covariance matrix is singular, one equation is deleted from the system for estimation. Maximum likelihood estimates of (34) are obtained using the Davidon-Fletcher-Powell algorithm as implemented in GQOPT

- There appears to be no obvious method for selecting the values for the elements of $\underline{\alpha}$.. Therefore, following Diewert and Wales (1988a, 1988b, 1993), we define $\alpha_i = (1/n)\ \forall\ i$ which clearly satisfies the restrictions in (36).

Results

- The unrestricted NQID-QSS does not have a negative semidefinite Antonelli matrix; there are two positive eigenvalues

- R^2 values are reported in Table 2, along with the optimized log likelihood values

- LR tests, reported in Table 3, indicate that the rank of the NQID-QSS model may be reduced to K=5 without a significant change in log likelihood values. This semiflexible model is the version of the NQID-QSS used to calculate welfare loss estimates in the following section

- Using the semiflexible model allows the number of estimated parameters to be decreased from 60 to 54

- An interesting question is whether or not the estimated a_i terms are significantly different from zero. If not, then quadratic terms no longer appear and the model reduces to the globally concave NQID model of Holt and Bishop

- As indicated at the bottom of Table 3, LR tests reveal that the null hypothesis of linear scale curves may be rejected at the 1% level for the unrestricted, restricted, and K=5 models. Therefore, there is strong evidence that linearity in scale is not a viable assumption in the present application

Welfare Losses

- Imposing curvature allows us to obtain consistent money-metric estimates of welfare losses occuring due to quantity reductions

- Money metric measures of (normalized) compensating variation (CV) and equivalent variation (EV) associated with a change in quantities from $\underline{q}^{0}$ to $\underline{q}^{1}$ are shown by Kim to equal:

(43) $$CV = m^0[D(u^0,\underline{q}^1) - D(u^0,\underline{q}^0)],$$

and

(44) $$EV = m^0[D(u^1,\underline{q}^1) - D(u^1,\underline{q}^0)],$$

where m^0 represents total expenditure before any change in quantities.

- Using equations (43) and (44) along with the estimated distance functions defined by (9) and (13)-(20) allows us to obtain estimates of welfare losses associated with an arbitrary reduction of the quantity landed for a particular species of fish.

- Welfare losses are calculated on a yearly basis with concavity being imposed at that particular year, as well as at the sample means. The exact welfare loss estimates for selected years are reported in Table 6.

- As would be expected, the CV and EV estimates are positive in every instance, showing that consumers are made unambiguously worse off following any restriction that reduces the quantity of a particular species landed. In general, differences between the CV and EV estimates are not very large.

Conclusions

- Found empirical support for including quadratic scale terms

- Empirical results also suggest that rank reduction may be used without a significant loss in fit

- Semiflexible model with K=5 used to obtain CV and EV estimates associated with an arbitrary ten percent reduction in quantity landed for individual species

- Appears that including quadratic terms offers an improvement in modeling systems in which quantities are taken as exogenous and may prove beneficial in future applications